Sincere and sophisticated players in an equal-income market

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Abstract

We study the simultaneous direct revelation mechanism associated with each equal-income competitive social choice function in the allocation of objects and money among sincere and strategic agents. Strategic agents take advantage of sincere agents. They non-cooperatively coordinate on the equal-income competitive allocations for the true preferences that are Pareto undominated for them within the set of equal-income competitive allocations. Sincere agents are protected to some extent, however. Their welfare is usually above their maximin payoff.

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1 Introduction

We consider the problem of allocating a set of \( n \) objects and an amount of money among \( n \) agents with unit demands and quasi-linear preferences. Examples are the dissolution of a partnership \([1]\) and the allocation of rooms and contributions to rent among roommates \([2]\). An equal-income competitive (eic) allocation is an allocation that can be sustained as a competitive equilibrium when each agent has an equal share of the aggregate income. An eic social choice function (eic-scf), our main object of study, selects an eic allocation for each possible preference profile. We characterize the non-cooperative equilibrium outcomes of the simultaneous direct revelation game associated with each eic-scf under the assumption that an exogenously

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determined set of agents are sincere, i.e., unconditionally report their true preferences, and this is common knowledge among the other agents, who are strategic. This allows us to make a normative assessment of these mechanisms. In some situations, it may be plausible that agents differ in their level of sophistication.\footnote{In the related problem of school choice, where parents report preferences on public schools, \cite{3} argue that parents’ sophistication is not homogeneous. Some parents may participate in extensive discussion of the best strategies given the mechanism adopted by a school district. Some other may report their true preferences without further thought. One can envision a similar situation in our environment. In experimental settings, recent studies have documented the propensity of some subjects to provide truthful reports in strategic communication games \cite{4,5}.} A policy maker may be interested in protecting the welfare of both the less sophisticated and those who blindly follow her instructions.\footnote{\cite{6} argue that this has been the case in school districts in England and Chicago.} Several relevant questions arise. What are the outcomes of the manipulation of these scfs under our behavioral assumptions? How do these outcomes compare to those when all agents are strategic? Are efficiency and equity obtained in some form? Do strategic agents take advantage of sincere agents? If so, to what extent? Could we differentiate eic-scfs in terms of the way they treat sincere agents?

Our main result, Theorem 1, allows us to answer all these questions. It states that, in a complete information setting, the limit Nash equilibrium outcomes \cite{7,8} of the direct revelation game associated with an eic-scf at some preference profile is the set of eic allocations, for true preferences, that are Pareto undominated for the strategic agents within the eic set—we discuss our choice of limit Nash equilibrium as the prediction for these games below. Eic allocations are Pareto efficient and envy-free, i.e., no agent prefers the allotment of any other agent to her own.\footnote{Eic allocations and envy-free allocations coincide in our model \cite{9}.} Thus, non-cooperative behavior in the manipulation of an eic-scf with sincere and strategic agents does not compromise these two properties. Since eic allocations are Pareto efficient, our theorem implies that when all agents are strategic, the set of non-cooperative outcomes of the manipulation of each eic-scf is exactly the eic set. Moreover, when some agents are sincere, these outcomes shrink to the “faces” of the eic set that are most favorable to strategic agents (see Sec. 2). In this sense, strategic agents take advantage of sincere agents.\footnote{This is in stark contrast to environments with indivisible goods without money. In marriage markets, even though there is a women-optimal stable matching, each stable matching is an equilibrium, for women, of the direct revelation game associated with the men-optimal stable scf when men truthfully report their preferences \cite{10}. Some equilibrium refinements obtain the women-optimal stable allocation as the unique outcome when the men-optimal stable scf is operated \cite{11}. By contrast, our results do not depend on any equilibrium refinement. In school choice problems the outcomes from the manipulation of the “Boston mechanism” when some parents are sincere and the other strategic, contains outcomes that are Pareto dominated for the strategic parents by another equilibrium outcome \cite{3}.} Eic-scfs provide a safety net for sincere agents, however. They guarantee them, at least, the minimum welfare among all eic allocations for the true preferences. This lower bound is usually above the agent’s maximin payoff.\footnote{There is a unique list of bundles, one for each object, among which an agent is indifferent and whose associated consumptions of money add up to the budget —this is the so called identical preferences lower bound \cite{12,13}; the agent gets her Maximin payoff at some eic allocation if there is an allocation of these bundles that is envy-free with respect to true preferences.} Thus, the surprising part of our result is that even though strategic agents may have available actions that drive a sincere agent to her maximin payoff,
non-cooperative behavior precludes them from doing so. Finally, the direct revelation games associated with the eic-sufs are all strategically equivalent. Thus, all eic-sufs are equivalent in the way they treat sincere agents.

Our second result, Proposition 1, furthers our understanding of the incentives of an agent to truthfully report her preferences when an eic-scf is operated. It states that generically the maximum payoff of an agent among all non-cooperative outcomes from the manipulation of an eic-scf is greater when she is strategic than when ceteris paribus she is sincere. This is surprising, because for each agent, say \( i \), there are eic-sufs for which true reports are a dominant strategy for agent \( i \) [14]. Thus, our proposition implies that when these eic-sufs are operated, agent \( i \)'s maximum utility when she unconditionally reports her truthful dominant strategy is generically lower than the utility she can achieve in some non-cooperative outcome that cannot be sustained by her truthful report.

Several papers have studied the non-cooperative outcomes of the direct revelation mechanisms associated with eic-sufs [15, 16, 13, 17, 18]. We depart from this literature in two ways. We are the first to analyze this problem in the presence of sincere agents. We are also the first to consider limit Nash equilibrium as the non-cooperative prediction of these games. We do so because the manipulation models in previous literature are either unsatisfactory or inadequate for our environment with sincere and sophisticated agents (see Sec. 3). As a byproduct of our main contribution, we determine the extent to which previous results are robust to the limit Nash equilibrium prediction. [15, 16, 13, 17, 18] uniformly concluded, for different non-cooperative predictions than ours, that when all agents are strategic and manipulate an eic-scf, they coordinate on the set of eic allocations for true preferences. This is even so when preferences are not quasi-linear [15, 17, 18]. We confirm that this basic result holds when preferences are quasi-linear and one considers limit Nash equilibria. However, and most remarkably, we construct an example in Sec. 5 showing that when preferences are not quasi-linear there can be limit Nash equilibrium outcomes of the manipulation of an eic-scf that are inefficient.

There is a growing interest in the economic implications of sincerity in strategic situations. [19, 20, 21] study implementation issues among agents who always take an action that implements the true-state social optimal outcome whenever it is available to them and it is a best response to the actions of the other agents. [22, 23] study strategic communication when misrepresenting information is costly. The main difference with this previous literature is that sincerity in our model is an exogenous characteristic of some agents, which reflects their lack of sophistication more than a cost to lie that can be overridden by self interest. In this respect our paper is closer to [3], which attempts to evaluate mechanisms under realistic behavioral assumptions, and to [24], which studies the role of character, an exogenous attribute of a candidate, in electoral competition.

The rest of this paper is organized as follows. Sec. 2 illustrates our results with an example. Sec. 3 introduces the model. Sec. 4 presents our results. Sec. 5 discusses the limits of our results and points to further research. For completeness, the Appendix states two results in previous literature that we use in our proofs.
2 An illustrating example

A set of roommates $N \equiv \{1, 2, 3\}$ collectively lease a house with rooms $A \equiv \{1, 2, 3\}$. Rent is $1,200. Each roommate is to receive one room and pay an amount of money for it. Payments should add up to $1,200, so there is no surplus or deficit.

<table>
<thead>
<tr>
<th>Roommate 1</th>
<th>Room 1</th>
<th>Room 2</th>
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<tr>
<td>roommate 1</td>
<td>$400</td>
<td>$400</td>
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<tr>
<td>roommate 2</td>
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<tr>
<td>roommate 3</td>
<td>$400</td>
<td>$700</td>
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Table 1: Valuations

Preferences are quasi-linear with valuations given by Table 1. That is, they are represented by the net value of the agent’s consumption. For example, if roommate 1 receives room 1 and pays $300 for it, her net value is 100.

There is a continuum of eic allocations for this problem. These allocations are Pareto efficient [9]. Thus, agent 1 receives room 1, agent 2 room 2, and agent 3 room 3 at each eic allocation (this is the unique assignment of rooms that maximizes aggregate value). Let $(r_i)_{i \in N}$ be a vector of eic payments of rent. Since eic allocations coincide with the set of envy-free allocations, i.e., those at which no agent prefers the consumption of another agent to her own consumption [9], these payments are those satisfying for each $i \neq j$ the linear inequality constraint $v^i_j - r_i \geq v^i_j - r_j$, where $v^i_j$ is agent $i$’s value of room $j$. Since $r_1 + r_2 + r_3 = 1,200$, we can solve this inequality system for $r_1$ and $r_2$ (Figure 1 (a)). The set of pairs $(r_1, r_2)$ associated with eic allocations is a polygonal with non-empty interior in $\mathbb{R}^2$ (Figure 1 (a)).

An eic-scf associates an eic allocation with each valuation profile. Our main result, Theorem 1, characterizes the limit Nash equilibrium outcomes of the direct revelation game of each eic-scf when a set of agents $T \subseteq N$ are sincere. Let $S \equiv N \setminus T$. These outcomes are the eic allocations for true preferences that are Pareto undominated for $S$ within the eic set. When no agent is sincere, the outcomes are the whole set of eic allocations for true preferences. When $T$ is non-empty, the outcomes are reduced to a subset of the “faces” of the eic set. For instance, if $T \equiv \{1\}$, these outcomes are those in the segment connecting $b$ and $c$ in Fig. 1 (b). This clearly illustrates that when only agent 1 is sincere, the other agents take advantage of her sincerity. One can also say that sincerity has a cost to agent 1. The maximum payoff that she gets if she is strategic, is strictly higher than the maximum payoff she gets if she is sincere. This is a generic phenomenon for all agents, independently of the eic-scf that is operated (Proposition 1). The extent to which strategic agents take advantage of sincere agents has a limit, however. No sincere agent here is forced to obtain her maximin payoff. One can easily prove that when preferences are quasi-linear an agent’s true valuations are her unique maximin strategy. Moreover, agent $i$’s maximin payoff is exactly that she would obtain should all agents report agent $i$’s valuations. In such a report, since all agents have the same preferences, for each eic

\footnote{Also observe that a sincere agent may receive a payoff that is greater than her worse eic payoff.}
Figure 1: (a) $A \equiv \{1, 2, 3\}$, $N \equiv \{1, 2, 3\}$, $r_1 + r_2 + r_3 = 1,200$, and valuations are given in Table 1. Let $l_{ij}$ be the linear inequality constraint $v_{ij}^i - r_i \geq v_{ij}^j - r_j$ where $v_{ij}^j$ is agent $i$’s value of room $j$. The figure displays each $l_{ij}$ with its label inside the half space satisfying the inequality constraint. The shaded area contains all $(r_1, r_2)$ such that the allocation where agent $i$ receives room $i$ and rent payments are $(r_1, r_2, 1,200 - r_1 - r_2)$ is envy-free and thus can be sustained as an eic equilibrium. Roommate 1’s indifference curves are vertical lines (she prefers lower $r_1$); roommate 2’s indifference curves are horizontal lines (she prefers lower $r_2$); roommate 3’s indifference curves are lines with constant $r_1 + r_2$ (she prefers lower $r_3$, i.e., higher $r_1 + r_2$). The indifference curve shown for each agent corresponds to her maximin payment. (b) equilibrium outcomes from the manipulation of any eic-scf when the set of agents $T$ is sincere.
A. The generic consumption bundle is \((x_a, \alpha)\). Agents have quasi-linear references. Let \(\mathcal{O}\) be the domain of quasi-linear normalized utility functions, i.e., functions of the form \((x_i, \alpha) \in \mathbb{R} \times A \rightarrow x_i + v_i^\alpha\), where \(v_i^\alpha \equiv (v_i^\alpha)_{\alpha \in \mathcal{A}} \in \mathbb{R}^A\) is such that \(\sum_{\alpha \in \mathcal{A}} v_i^\alpha = 0\). The utility profile is \(u \equiv (u_i)_{i \in \mathcal{N}} \in \mathcal{O}^N\). For each \(i \in \mathcal{N}\) and each \(u'_i \in \mathcal{O}\), the profile \((u_{-i}, u'_i)\) is obtained from \(u\) by replacing \(u_i\) by \(u'_i\). For each \(K \subseteq \mathcal{N}\), \(u_K\) is the subprofile \((u_i)_{i \in K}\).

An allocation is a pair \(z \equiv (x, \mu)\) where \(x \equiv (x_a)_{a \in \mathcal{A}} \in \mathbb{R}^A\) is such that \(\sum_{a \in \mathcal{A}} x_a = M\) and \(\mu \equiv (\mu_i)_{i \in \mathcal{N}} \in \mathcal{A}^N\) is such that for any two different agents \(i\) and \(j\), \(\mu_i \neq \mu_j\). Notice that consumptions of money are indexed by objects in vector \(x\). Thus, \(x_a\) is the consumption of money of the agent who receives \(\alpha\) at \(z\). In order to simplify notation, whenever convenient, we denote agent \(i\)'s consumption of money at \(z\) by \(x_i\). Agent \(i\)'s allotment at \(z\) is then \(z_i \equiv (x_i, \mu_i) = (x_i, \mu_i)\). Let \(Z\) be the set of all allocations. Given \([z, z'] \subseteq Z\) and \(S \subseteq \mathcal{N}\), we say that \(z\) Pareto dominates \(z'\) for \(S\) at \(u\) if for each \(i \in S\), \(u_i(z_i) \geq u_i(z'_i)\) and for at least one \(j \in S\), \(u_j(z_j) > u_j(z'_j)\).

An allocation \(z \in Z\) is an eic allocation for \(u \in \mathcal{O}^N\) if there is a price vector \(p \in \mathbb{R}^A\), such that for each \(i \in N\), \(z_i\) maximizes \(u_i\) in the equal-income budget set, i.e., \(\{(x_a, \alpha) : a \in A, x_a \leq M, p_{\alpha} = 1\} \), where \(I \equiv M + \sum_{a \in A} p_{\alpha}\). We denote the set of these allocations for \(u\) by \(\mathcal{W}_e(u)\). It is well known that \(z \in \mathcal{W}_e(u)\) if and only if it is envy-free for \(u\), i.e., for each \(i, j \in \mathcal{N}\), \(u_i(z_i) \geq u_j(z_i)\).

### 3.2 Social choice functions and their simultaneous direct revelation game form

An scf, generically denoted by \(f\), associates with each \(u \in \mathcal{O}^N\) an allocation \(f(u) \in Z\). An scf \(f\) is an eic-scf if for each \(u \in \mathcal{O}^N\), \(f(u) \in \mathcal{W}_e(u)\).

We study the manipulation of an scf, say \(f\), when some agents are sincere, this is common knowledge, and the other agents are strategic. For this purpose, we consider the simultaneous direct revelation game associated with \(f\) when a group of agents \(T \subseteq \mathcal{N}\) is sincere, true preferences are represented by \(u \in \mathcal{O}^N\), and the set of admissible utility functions is \(\mathcal{O}\). We assume complete information. Thus, this is a game for the set of players \(S \equiv \mathcal{N} \setminus T\) in which each player's strategy space is \(\mathcal{O}\), the outcome function is \(f(u_T, \cdot)\), and agents' preferences on outcomes are represented by \(u_S\). We denote this game by \((S, \mathcal{O}^S, f(u_T, \cdot), u_S)\).

The set of pure strategy Nash equilibria of the game associated with an eic-scf may be empty. For instance, suppose that there are only two agents. Then, agents' reports must be equal in each pure strategy Nash equilibrium of the direct revelation game associated with any eic-scf. When reports are equal, there is exactly two eic allocations, which are welfare equivalent for both agents. Thus, an eic-scf acts as a tie breaker in these profiles. Similarly to a Bertrand competition game, the dis-

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Notice that object prices are not restricted to be non-negative, so agents may be compensated for receiving an object. The price of money is normalized to 1.

\(^8\)\(\mathcal{W}_e(u)\) is nonempty \([25, 26]\); moreover, since there are as many agents as objects, these allocations are “efficient,” i.e., Pareto undominated in \(Z\) \([9]\).

\(^9\)Envy-free allocations were introduced by \([27]\) and have been central in the study of fair allocation. See \([28]\) for a survey.
continuity induced by the tie-breaker induces non-existence of pure strategy Nash equilibria for most profiles.

The solutions in previous literature to the issue of non-existence of pure-strategy Nash equilibria in the manipulation of eic-scfs are either unsatisfactory or inadequate for our model with sincere and strategic agents. A social choice correspondence (scc) $\varphi$ associates with each $u \in \mathcal{Z}^N$ a set of allocations $\varphi(u) \subseteq \mathbb{Z}$. [29, 30] avoided the non-existence of equilibria of the manipulation of scfs by generalizing the notion of a game-form associated with an scf to a “quasi-game-form” associated with an scc $\varphi$. In such a quasi-game-form the outcomes are not completely determined by an scf as in the standard mechanism design framework. Instead, they are restricted to be one of the $\varphi$-optimal allocations for the reported profile. A quasi-game-form induces a direct revelation quasi-game, for each preference profile, in which each agent reports her preferences and the outcome is restricted by $\varphi$. A quasi-equilibrium of this quasi-game is a pair $(u^*, z)$, such that $z \in f(u^*)$ and no agent, who envisions a best-case scenario at an alternative report, can gain with respect to $z$ by deviating from $u^*$. [15, 13] applied this approach to the manipulation of eic-scfs in our environment with indivisible goods and money and showed that for each essentially single-valued and Pareto indifferent eic-scc and each $u$ the set of quasi-equilibria of the respective quasi-game is the set of eic allocations for $u$. This approach is unsatisfactory from a mechanism design perspective, for predictions depend on the coarse definition of the allocation process. Note that the direct revelation quasi-game-form associated with an scf is actually its standard direct revelation game-form, and its quasi-equilibria are actually pure strategy Nash equilibria. Thus, quasi-equilibria are only guaranteed to exist for eic-scfs that are multi-valued. Thus, an arbitrator relying on this theory would need to leave ties unresolved and expect agents will not only play mutual best responses, but also coordinate —somehow— on an outcome that sustains those best responses. If the arbitrator explicitly breaks the ties, the non-cooperative prediction may be empty. In order to bypass this problem, [31, 17] enlarge agents’ strategy space in the direct revelation game associated with an scc and provide explicit actions that allow agents to endogenously break ties when the scc is multivalued for the profile of reports. This manipulation model is inadequate for our purposes, for it is not clear what a truthful report is in an action space different from an agent’s admissible domain of preferences. Finally, [18] observe that when the agents’ domain of admissible utilities may contain multiple representations of a single preference or utilities that are not quasi-linear, the direct revelation game-forms associated with some eic-scfs may posses non-empty pure strategy Nash equilibrium outcome correspondences. This is of no help in our model where admissible preferences are quasi-linear and utilities are normalized.

An intuitive prediction to the direct revelation games induced by an eic-scf, which is not affected by the aforementioned issues, is the so-called limit equilibria [7, 8]. Let $\varepsilon > 0$. A profile $u^\varepsilon \in \mathcal{Z}^N$ is an $\varepsilon$-equilibrium of $(S, \mathcal{Z}^S, f(u_T^\cdot), u_S)$ if no agent can gain more than $\varepsilon$ in utility by choosing a different action in $\mathcal{Z}$. An allocation $z$ is a limit Nash equilibrium outcome of $(S, \mathcal{Z}^S, f(u_T^\cdot), u_S)$ if there is a sequence of $\varepsilon$-equilibrium outcomes that converges to $z$ as $\varepsilon$ vanishes. We denote this set by $\mathcal{O}(S, \mathcal{Z}^S, f(u_T^\cdot), u_S)$.
In our model with money, we can interpret limit equilibrium outcomes as those that can be sustained as the limit of a sequence of near pure-strategy equilibrium outcomes when one discretizes money consumptions and the minimal increment of money becomes negligible. In particular, this implies that the limit equilibrium prediction is ordinal, i.e., it does not depend on our choice of $u$ among the continuous functions representing preferences.

**Remark 1** (Utility-free characterization of limit equilibria). Let $T \subseteq N$, $S \equiv N \setminus T$, $f$ an scf, and $u \in \mathcal{Q}^N$. A utility-free $\varepsilon$-Nash equilibrium of $(S, \mathcal{Q}^S, f(u_T \cdot), u_S)$ is a profile $u_S^\varepsilon$ for which no agent can obtain, by changing her report, an allocation that is preferred to the bundle obtained by adding $\varepsilon$ of money to her consumption at $u_S^\varepsilon$. An allocation is a utility-free limit Nash equilibrium outcome of $(S, \mathcal{Q}^S, f(u_T \cdot), u_S)$ if it is the limit as $\varepsilon \to 0$ of a sequence of utility-free $\varepsilon$-Nash equilibria. It is straightforward to check that since $u_S$s are continuous, $\partial(S, \mathcal{Q}^S, f(u_T \cdot), u_S)$ coincides with the set of utility-free limit Nash equilibrium outcomes of $(S, \mathcal{Q}^S, f(u_T \cdot), u_S)$.

## 4 Manipulation of an equal-income market

When manipulating an eic-scf, strategic agents non-cooperatively coordinate on the eic allocations for true preferences that are Pareto undominated for them in the set of eic allocations.

**Theorem 1.** Let $f$ be an eic-scf, $u \in \mathcal{Q}^N$, $T \subseteq N$, and $S \equiv N \setminus T$. Then,

(i) The set $\partial(S, \mathcal{Q}^S, f(u_T \cdot), u_S)$ is non-empty.

(ii) If $z \in \partial(S, \mathcal{Q}^S, f(u_T \cdot), u_S)$, then $z \in W_e(u)$ and there is no allocation in $W_e(u)$ that Pareto dominates $z$ for $S$ at $u$.

(iii) If $z \in W_e(u)$ and there is no allocation in $W_e(u)$ that Pareto dominates $z$ for $S$ at $u$, then there is $\tilde{z} \in \partial(S, \mathcal{Q}^S, f(u_T \cdot), u_S)$ such that for each $i \in N$, $u_i(\tilde{z}) = u_i(z)$.

The proof of Theorem 1 follows from four lemmas of independent interest. The first three lemmas provide an expedite test, or algorithm, to determine whether an allocation is a limit Nash equilibrium outcome from the manipulation of an eic-scf when some agents are sincere. They state necessary and sufficient conditions in terms of the following binary relation.

**Definition:** for each pair $i, j \subseteq N$, $i \succeq (u, z)$ if there is a list of agents $i_1, i_2, \ldots, i_k$ such that $u_i(z_{i_1}) = u_i(z_{i_2}), \ldots, u_{i_k}(z_{i_k}) = u_{i_k}(z_{i_k})$, $i_1 = i$, and $i_k = j$.\(^{10}\)

Let $z$ be an allocation and $u \in \mathcal{Q}^N$. Intuitively, agent $i$ dominates agent $j$, in terms of $\succeq (u, z)$, when agent $i$’s consumption at $z$ can be connected to that of $u$.

\(^{10}\)The binary relation $\succeq (u, z)$ was introduced, in its equivalent form for the set of objects, by [32]. The use of this binary relation for the study of the manipulation of eic-scf was pioneered by [17]. Following this approach, [14, 33] and [18] characterize the situations in which an agent is able to manipulate any eic-scf. This characterization is an important tool for our analysis. For completeness, we state it in the Appendix (Theorem 2).
Let $f$ be an $\varepsilon$-scf, $u \in \mathcal{Q}^N$, $T \subseteq N$, and $S \equiv N \setminus T$. Then,

$$\Theta(S, \mathcal{Q}^S, f(u_T, \cdot), u_S) \subseteq W_c(u).$$

**Proof.** Let $z \in \Theta(S, \mathcal{Q}^S, f(u_T, \cdot), u_S)$. We prove that $z \in W_c(u)$. Suppose by contradiction that there is $(i, j) \subseteq N$ such that $u_i(z_j) > u_i(z_i)$. Suppose without loss of generality that $z_j$ is agent $i$’s preferred bundle in $\{z_k\}_{k \in N}$. Let $u^\varepsilon \equiv (x^\varepsilon, \mu^\varepsilon) \equiv (x^\varepsilon, \mu^\varepsilon)$, such that as $\varepsilon \to 0$, $z^\varepsilon \to z$. We can suppose without loss of generality that for each $\varepsilon$, $\mu^\varepsilon = \mu$. Since as $\varepsilon \to 0$, $z^\varepsilon \to z$, and preferences are continuous, then for each $k \in T$ and each $j \in N$, $u_k(z_k) \geq u_k(z_j)$. Thus, $i \in S$. Let $\tilde{u}_i \equiv (u_{i_T}, u_{i_S}).$ Let $\delta \equiv \frac{1}{\varepsilon} |u_i(z_j) - u_i(z_i)|$. Since as $\varepsilon \to 0$, $z^\varepsilon \to z$, there is a $\eta > 0$ such that for each $\varepsilon < \eta$, $|u_i(z_j^\varepsilon) - u_i(z_i)| \leq \frac{\delta}{2}$ and $|u_i(z^\varepsilon_i) - u_i(z_i)| \leq \frac{\delta}{2}$. Thus, $|u_i(z_j^\varepsilon) - u_i(z_i^\varepsilon)| \leq \frac{\delta}{2}$. Suppose first that $j \geq (\tilde{u}_i, z^\varepsilon_i)$. Then, there is a way to reshuffle consumption bundles at $z^\varepsilon$ so that agent $i$ receives $\mu_j$ and the resulting allocation belongs to $W_c(\tilde{u}_i)$ (just reshuffle along the chain that implies $j \geq (\tilde{u}_i, z^\varepsilon_i)$). By Theorem 2 (see Appendix), there is $u^\varepsilon \in \mathcal{Q}$ such that $u_i(f(u_T, u^\varepsilon, \mathcal{Q}^S)) \geq u_i(z_j^\varepsilon) - \delta = u_i(z^\varepsilon_i) + (u_i(z_i^\varepsilon) - u_i(z^\varepsilon_i)) - \delta \geq u_i(z^\varepsilon_i) + 5\delta$. Thus, if $\varepsilon < \min\{\delta, \eta\}$, $(u_{i_T}, u_{i_S})$ is not an $\varepsilon$-equilibrium of $(S, \mathcal{Q}^S, f(u_T, \cdot), u_S)$. This is a contradiction. Now, let $N' \equiv \{k : k \geq (\tilde{u}_i, z^\varepsilon_i)\}$. Suppose that $j \in N \setminus N'$. Let $t \equiv \min_{k \in N', n} \tilde{u}'_i(z^\varepsilon_k) - \tilde{u}'_i(z^\varepsilon_i)$. Consider the allocation obtained from $z^\varepsilon$, without reshuffling objects, by extracting from each agent in $N \setminus N'$ an amount of money $t\frac{|N'|}{n}$ and distributing the collected $t\frac{|N'|}{n}u_j$ in equal parts among the agents in $N'$. Thus each agent in $N'$ receives $t\frac{|N'|}{n}u_j$. Let $z'$ be this allocation. One can easily see that $N' \subseteq \{k : k \geq (\tilde{u}_i, z')\}$ and for each $k \neq i$ and each $l \in N$, $u_k(z^\varepsilon_k) \geq u_k(z^\varepsilon_i)$. If $j \notin \{k : k \geq (\tilde{u}_i, z')\}$, repeat the process and extract a common amount from the agents in $N \setminus \{k : k \geq (\tilde{u}_i, z')\}$ and distribute it in equal parts among the other agents so that at the resulting allocation, say $z''$, there is no envy from an agent different from $i$ to another agent and the set $\{k : k \geq (\tilde{u}_i, z'')\}$ strictly contains $\{k : k \geq (\tilde{u}_i, z')\}$. By repeating this process at most $n - 1$ times, one arrives at an allocation $\bar{z}$ such that $j \in \{k : k \geq (\tilde{u}_i, z')\}$; for each $k \neq i$ and each $l \in N$, $u_k(\bar{z}_k) \geq u_k(\bar{z}_i)$; and $u_i(\bar{z}_j) = u_i(\bar{z}_i) + \frac{\mu_j}{\mu_i}(u_i(\bar{z}_j) - u_i(\bar{z}_i))$, because agent $j$ contributes with money and agent $i$ receives money in each step in the algorithm that produces $\bar{z}$. Let $l \in N$ be such that $\bar{z}_j$ is one of the best bundles for $u_l$ in $\{z_k\}_{k \in N}$. Since $z_j$ is agent $u_i$’s preferred bundle in $\{z_k\}_{k \in N}$, then we can assume without loss of generality that $l \in \{k : k \geq (\tilde{u}_i, z')\}$. Let $z^* \equiv (x^*, \mu^*)$ be the allocation obtained by reshuffling bundles along the chain that implies $l \geq (\tilde{u}_i, \bar{z})$ so that $\mu^*_l = \mu_l$. We claim that $u_i(z^*) \geq u_i(z^*_j) - \frac{\mu_j}{\mu_i}(u_i(z^*_j) - u_i(z^*_i))$. There are two cases.

**Case 1:** $u_i(z^*_j) \geq u_i(z^*_i) - \frac{\mu_j}{\mu_i}(u_i(z^*_j) - u_i(z^*_i))$. Thus, $u_i(z^*_j) \geq u_i(z^*_i) + \frac{1}{\mu_i}(u_i(z^*_j) - u_i(z^*_i))$. **Case 2:** $u_i(z^*_j) < u_i(z^*_i) - \frac{\mu_j}{\mu_i}(u_i(z^*_j) - u_i(z^*_i))$. Recall that $u_i(z^*_j) - u_i(z^*_i) \geq \frac{1}{\mu_i}(u_i(z^*_j) - u_i(z^*_i))$. 


Thus, \( u_i(\tilde{z}_j) \geq u_i(z_j^*) + \frac{1}{n-1} (u_i(z_j^*) - u_i(z_j^*)) \) and \( u_i(z_j^*) \geq u_i(z_j^*) + \frac{1}{n-1} (u_i(z_j^*) - u(z_j^*)) \).

Now, \( u_i(z_j^*) \geq u_i(z_j^*) + \frac{1}{n-1} (u_i(z_j^*) - u_i(z_j^*)) \geq u_i(z_j^*) + \frac{1}{n-1} (u_i(z_j) - u_i(z_i) - 2\delta) \geq u_i(z_j^*) + \frac{1}{n-1} (7(n-1)\delta - 2\delta) > u_i(z_j^*) + \delta. \) Let \( \delta' \equiv u_i(z_j^*) - u_i(z_j^*) + \delta \).

Let \( u_i^* \) be such that the maximizer of \( u_i^* \) in \( \{z_k^*\}_{k \in \mathbb{N}} \) is \( z_j^* \) and for each \((x, a)\) such that \( \alpha \neq \mu^* \) and \( u_i^*(x, a) = u_i(z_j^*) \), \( u_i(x, a) > u_i(z_j^*) \). Then, \( z^* \in W_i(u_T, u_i^*, u^{S}_{S(i)}[i]). \) By Theorem 2, there is \( u_i^* \in \mathcal{D}' \) such that \( u_i^*(f_i(u_T, u_i^*, u^{S}_{S(i)}[i])) \geq u_i(z_j^*) - \delta'. \) Thus, \( u_i(f_i(u_T, u_i^*, u^{S}_{S(i)}[i])) \geq u_i(z_j^*) + \delta. \) Thus, if \( \epsilon < \min(\delta, \eta), (u_T, u^*_S) \) is not an \( \epsilon \)-equilibrium of \((S, \mathcal{D}', f(u_T, \cdot), u_S). \) This is a contradiction.

\[ \square \]

**Lemma 2.** Let \( f \) be an eic-sc. \( u \in \mathcal{D}^N, T \subseteq N, \) and \( S \equiv \mathbb{N} \setminus T. \) If \( \mathcal{O}(S, \mathcal{D}^S, f(u_T, \cdot), u_S), \) then for each \( i \in T \) there is \( \epsilon \geq (u, z) j. \)

**Proof of Lemma 2.** Let \( z \equiv (x, \mu) \in \mathcal{O}(S, \mathcal{D}^S, f(u_T, \cdot), u_S) \) and \( u^*_j \) be a sequence of \( \epsilon \)-equilibria of \((S, \mathcal{D}^S, f(u_T, \cdot), u_S) \) such that as \( \epsilon \to 0, z^* \to z. \) Let \( u^* \equiv (u_T, u^*_S). \) \( z^* \equiv (x^*, \mu^*) = f(u^*) \) be the corresponding outcomes in the sequence. We can suppose without loss of generality that for each \( \epsilon, \mu^* = \mu. \) By Lemma 1, \( z \in W_i(u). \)

Let \( i \in T. \) We claim that there is \( j \in S \) such that \( i \geq (u, z) j. \) Suppose by contradiction that there is no \( j \in S \) such that \( i \geq (u, z) j. \) Let \( j \in S \) and \( N' \equiv \{k : k \geq (u, z) j \} \cup S. \) Thus, \( j \in N' \) and \( i \in \{k : i \geq (u, z) j \} \subseteq N' \setminus N' \subseteq T. \)

Let \( t \equiv \min_{k \in \mathbb{N} \cap N', \mu^* \in \mathcal{D}^N_{S(i)}} u^*_i(z_k) - u^*_i(z_i) \) and \( t' \equiv \min_{k \in \mathbb{N} \cap N', \mu^* \in \mathcal{D}^N_{S(i)}} u^*_i(z_k). \) Since \( N' \subseteq T, t \equiv \min_{k \in \mathbb{N} \cap N', \mu^* \in \mathcal{D}^N_{S(i)}} u^*_i(z_k) - u^*_i(z_i) \) and \( t' \equiv \min_{k \in \mathbb{N} \cap N', \mu^* \in \mathcal{D}^N_{S(i)}} u^*_i(z_k) - u^*_i(z_i). \) Let \( \delta = \frac{1}{2} \) and \( \eta > 0 \) be such that for each \( \epsilon < \eta, \max_{\alpha \in A} |x_\alpha - x^*_\alpha| < \delta. \) Thus, for each \( \epsilon < \eta, t' \geq \epsilon t. \) Let \( z^* \) be the allocation obtained from \( z^* \), without reshuffling objects, by extracting \( \frac{N(N')}{n} t' \) from each agent in \( N \setminus N' \) and distributing the collected \( \frac{N(N')}{n} t' \) in equal parts among the agents in \( N'. \) Thus, each agent in \( N' \) receives an additional \( \frac{N(N')}{n} t' \). Let \( \tilde{u}^*_j \) be such that the maximizer of \( \tilde{u}^*_j \) in \( \{z_k^*\}_{k \in \mathbb{N}} \) is \( z_j^* \) and for each \((x_\alpha, \alpha)\) such that \( \alpha \neq \mu^*_j \) and \( \tilde{u}^*_j(x_\alpha, \alpha) = \tilde{u}^*_j(z_j^*), \) \( u_i(x_\alpha, \alpha) > \tilde{u}^*_j(z_j^*) \).

Let \( \tilde{u}^* = (u_T, \tilde{u}^*_j, u^*_S(i)). \) Since preferences are quasi-linear, one can easily verify that \( z^* \in W_i(\tilde{u}^*). \) Moreover, \( \tilde{u}^*_j(z_j^*) + \frac{N(N')}{n} t' \geq \tilde{u}^*_j(z_j^*) + \frac{N(N')}{n} t' \geq \tilde{u}^*_j(z_j^*) + \frac{1}{2n} t. \) By Theorem 2, there is \( u^*_j \in \mathcal{D} \) such that \( \tilde{u}^*_j(f_j(u_T, u^*_j, u^*_{S(i)}[i])) \geq \tilde{u}^*_j(z_j^*) + \frac{1}{2n} t. \) Thus, \( u^*_j(f_j(u_T, u^*_j, u^*_{S(i)}[i])) \geq u_j(z_j^*) + \frac{1}{4n} t. \) Thus, if \( \epsilon < \min \{\frac{1}{4n}, \eta\}, u^*_S \) is not an \( \epsilon \)-equilibrium of \((S, \mathcal{D}'^S, f(u_T, \cdot), u_S). \) This is a contradiction.

\[ \square \]

Our third lemma states that, in welfare terms, the converse to the joint implications of Lemmas 1 and 2 holds.

**Lemma 3.** Let \( f \) be an eic-sc. \( u \in \mathcal{D}^N, T \subseteq N, \) and \( S \equiv \mathbb{N} \setminus T. \) Let \( \tilde{z} \in W_i(u) \) be such that for each \( i \in T \) there is \( \delta \geq (u, \tilde{z}) j. \) Then, there is \( z \in \mathcal{O}(S, \mathcal{D}^S, f(u_T, \cdot), u_S) \) such that for each \( i \in T, u_i(\tilde{z}) = u_i(z_i) \) and for each \( j \in S, z_j = \tilde{z}_j. \)

**Proof.** Let \( \tilde{z} \equiv (\tilde{x}, \tilde{\mu}) \in W_i(u) \) be such that for each \( i \in T \) there is \( \delta \geq (u, \tilde{z}) j. \) Let \( \epsilon > 0. \) Let \( u^*_S \in \mathcal{D}^S \) be the utility profile such that for each \( j \in S \) and each \( k \neq j, u^*_S(\tilde{z}_k) = u^*_j(\tilde{x}_k + \epsilon, \tilde{\mu}_k). \) Let \( z^* \equiv (x^*, \mu^*) = f(u_T, u^*_S). \) Observe that \( \{z^*, \tilde{z}\} \subseteq W_i(u_T, u^*_S). \)
Step 1: For each $j \in N$, $x_i^j \leq \bar{x}_{\mu_i^j} + \varepsilon$. Suppose by contradiction that there is $j \in N$ such that $x_i^j > \bar{x}_{\mu_i^j} + \varepsilon$. By Lemma 5 (see Appendix), $x_i^j > \bar{x}_{\hat{\mu}_i}$. Since $z^\varepsilon \in \mathcal{W}_c(u_T, u_i^j)$, for each $k \in S \setminus \{j\}$, $u_k^j(z_i^j) \geq u_k^j(z_i^\varepsilon)$. By Lemma 5, for each $k \in S \setminus \{j\}$, $x_i^k > \bar{x}_{\hat{\mu}_i}$. Thus, for each $k \in S \setminus \{j\}$, $x_i^k > \bar{x}_{\hat{\mu}_i}$. Let $i \in T$. There is $j \in S$ such that $i \in (u, z)_j$. Thus, there is a list of agents $i_1, \ldots, i_m$ such that $i_1 = i$, $i_m \in S$, and for each $s = 1, \ldots, m - 1$, $i_s \in T$ and $u_i(z_{i_s}) = u_i(z_{i_{s+1}})$. Thus, $u_{i_{s-1}}(z_{i_{s-1}}) \geq u_{i_{s-1}}(x_{i_{s-1}}^q, \hat{\nu}_{i_{s-1}}^q), u_{i_{s}}(\hat{\mu}_{i_{s}}^q) > u_{i_{s}}(z_{i_{s+1}})$. By Lemma 5, $x_i^q > \bar{x}_{\hat{\mu}_{i_{s-1}}}$. The recessive argument shows that $x_i^q > \bar{x}_{\hat{\mu}_i}$. Thus, $\sum_{k \in N} x_i^k > \sum_{k \in N} \bar{x}_k$. This is a contradiction.

Step 2: For each $j \in S \setminus \{i\}$, $x_j^i \geq \bar{x}_j - \frac{\varepsilon}{n}$. Suppose by contradiction that there is $j \in S \setminus \{i\}$ such that $x_j^i \leq \bar{x}_j - \frac{\varepsilon}{n}$. Let $z' = (x', \hat{\mu})$ be such that for each $k \neq j$, $x_k' = \bar{x}_k + \frac{\varepsilon}{n}$ and $x_j' = \bar{x}_j - \frac{\varepsilon}{n}$. Then, $z' \in \mathcal{Z}$ and $u_j^i$ assigns equal values to all bundles $\{z_j^i\}_{k \in N}$. Since $z^\varepsilon \in \mathcal{W}_c(u_T, u_i^j)$ and $x_j^i \neq \bar{x}_j - \frac{\varepsilon}{n}$, for otherwise there is $j \neq i$ such that $x_j^i > x_j^i$ and consequently $u_j^i(z_j^i) > u_j^i(z_j^e)$. Thus, $x_j^i > x_j^\varepsilon$. By Lemma 5, $u_i(z_j^i) > u_i(z_j^e)$. Thus, $x_j^i > x_j^\varepsilon + \varepsilon$. This contradicts our claim in Step 1.

Step 3: For each $j \in N$, $x_i^j \leq \bar{x}_i - \frac{\varepsilon}{n}$. Suppose by contradiction that for some $i \in N$, $x_i^j > \bar{x}_i - \frac{\varepsilon}{n}$. Suppose first that $i \in S$. By Step 2, $\mu_i^j = \hat{\mu}_i$. Since $z^\varepsilon \in \mathcal{W}_c(u_T, u_i^j)$, for each $k \neq i$, $u_k(z_i^j) \geq u_k(z_i^e)$. Thus, for each $k \neq i$, $x_k^i < \bar{x}_k + \frac{\varepsilon}{n}$. Thus, $\sum_{k \in N} x_i^k < \sum_{k \in N} \bar{x}_k$. This is a contradiction. Suppose now that $i \in T$. Let $N_1 = \{k \in N : x_k^i < \bar{x}_k - \frac{\varepsilon}{n}\}$ and $O_1 = \{a \in A : x_a^i < \bar{x}_a - \frac{\varepsilon}{n}\}$. We claim that $\mu$ is a bijection between $N_1$ and $O_1$. Clearly, $i \in N_1 \neq \emptyset$. We claim that $\bar{x}_k - \frac{\varepsilon}{n}$. By Lemma 5, $x_k^i \geq \bar{x}_k - \frac{\varepsilon}{n}$. Let $j \in N_1$. If $j \in N$ is such that $x_j^i(x_j^i, \mu^j) \leq x_j^i(\bar{x}_j, \mu^i)$, we have that $u_k(x_j^i, \mu^j) > x_k^i(\bar{x}_k, \mu^j)$. This contradicts $z^\varepsilon \in \mathcal{W}_c(u_T, u_i^j)$. Now, let $j \in N_1$. If $j \in N$ is such that $u_k(\bar{x}_j, \mu^i) \leq u_k(x_j^i, \mu^j)$, we have that $u_k(x_j^i, \mu^i) \leq u_k(\bar{x}_j, \mu^i)$. Let $j \in S$ be such that $i \epsilon (u, z)_j$. Thus, there is a list of agents $i_1, \ldots, i_m$ such that $i_1 = i$, $i_m \in S$, and for each $s = 1, \ldots, m - 1$, $i_s \in T$ and $u_i(z_{i_s}) = u_{i_s}(z_{i_{s+1}})$. Thus, $i_m \in N_1 \subseteq T$. This is a contradiction.

Step 4: $u_i^j$ is a $2\varepsilon$-equilibrium of $(S, \mathcal{O}^j, f(u_T, \cdot), u_S)$. Let $i \in S$, $u_i^j \in \mathcal{O}$, and $z^\varepsilon = f(u_T, u_i^j, u_S^i(j))$. By Theorem 2, $u_i(z_i^j) \leq u_i(z_i^e)$ where $z^\varepsilon \in \arg\max \{u_i(z_i) : z \in \mathcal{W}_c(u_T, u_i, u_S^i[j])\}$. Note that $\{z^\varepsilon, z^e \subseteq \mathcal{W}_c(u_T, u_i, u_S^i[j])\}$. By the same argument in Step 1, $x_i^j \leq \bar{x}_j + \varepsilon$. Thus, $u_i(z_i^j) \leq u_i(x_i^j, \mu_i^j) + \varepsilon$. Since $z \in \mathcal{W}_c(u_T, u_i^j)$, $u_i(x_i^j, \mu_i^j) \leq u_i(\bar{x}_i, \hat{\mu}_i)$. By Steps 2 and 3, $u_i(z_i, \hat{\mu}_i) \leq u_i(x_i^j, \mu_i^j) + \frac{\varepsilon}{n}$. Thus, $u_i(z_i^j) \leq u_i(x_i^j, \mu_i^j) + 2\varepsilon$. Thus, $u_i(z_i^e) \leq u_i(x_i^j, \mu_i^j) + 2\varepsilon$.

Step 5: Concludes. Consider a sequence $\varepsilon \to 0$ and the corresponding sequences $u^\varepsilon$ and $z^\varepsilon \equiv (x^\varepsilon, \mu^\varepsilon)$. Since there are finitely many objects, one can extract a subsequence in which each agent receives the same object in each $z^\varepsilon$. By Steps 1 and 3, $z^\varepsilon$ is a convergent sequence as $\varepsilon \to 0$. Let $z \equiv (x, \mu)$ be the limit of this sequence. Thus, $z$ is a limit equilibrium of $(S, \mathcal{O}_s, f(u_T, \cdot), u_S)$. By Step 2, for each $i \in S, \hat{\mu}_i = \hat{\mu}_i$. By Steps 1 and 3, for each $\alpha \in A$, $x_\alpha = x_\alpha$. Since preferences are continuos, for each $i \in T$, and each $k \neq i$, $u_i(z_i) \geq u_i(z_k)$. Thus, $z \in \mathcal{W}_c(u)$. By Lemma 5, for each $i \in N$, $u_i(z_i) = u_i(z_i)$. 

Lemmas 1-3 characterize the limit Nash equilibrium outcomes of the manipula-
tion of an eic-scf with sincere and strategic agents, as, essentially, the eic allocations with respect to true preferences at which each agent can be connected to a strategic agent through a chain of indifferences at the consumptions in the allocation. It turns out that this property characterizes precisely eic allocations that are Pareto undominated for strategic agents in the eic set.

**Lemma 4.** Let \( T \subseteq N, S \equiv N \setminus T, u \in \mathcal{Q}, \) and \( z \in W_c(u) \). Then there is no \( z' \in W_c(u) \) that Pareto undominales \( z \) for \( S \) at \( u \), if and only if, for each \( i \in T \) there is \( j \in S \) such that \( i \geq (u, z) j \).

**Proof.** Suppose that \( \hat{z} \in W_c(u) \) and there is no \( z' \in W_c(u) \) such that for each \( i \in S \), \( u_i(z'_i) \geq u_i(\hat{z}_i) \) and for some \( j \in S, u_i(z'_j) > u_i(\hat{z}_j) \). We claim that for each \( i \in T \) there is \( j \in S \) such that \( i \geq (u, z) j \). Suppose by contradiction that there is \( i \in T \) such that for each \( j \in S \) it is not true that \( i \geq (u, z) j \). Let \( M \equiv \{ j \in N : i \geq (u, z) j \} \subseteq T \). One can construct an allocation \( z' \in W_c(u) \) that is preferred by all agents in \( N \setminus M \) by extracting some money from the agents in \( M \) and distributing it among the agents in \( N \setminus M \). This is a contradiction.

Suppose that \( z \in W_c(u) \) and for each \( i \in T \) there is \( j \in S \) such that \( i \geq (u, z) j \). We claim that there is no \( z' \in W_c(u) \) such that for each \( i \in S, u_i(z'_i) \geq u_i(z_i) \) and for some \( j \in S, u_i(z'_j) > u_i(z_j) \). Let \( z' \in W_c(u) \) be such that for each \( i \in S, u_i(z'_i) \geq u_i(z_i) \). By Lemma 5, for each \( i \in S, x'_i \geq x_i \). Let \( i \in T \). There is \( j \in S \) such that \( i \geq (u, z) j \). Since \( x'_i \geq x_i \), then an argument as that in Step 1 in the proof of Lemma 3 proves that \( u_i(z'_i) \geq u_i(z_i) \). By Lemma 5, \( x'_i \geq x_i \). Thus, \( x' = x \). Since \( z \in W_c(u) \), then for each \( i \in S, u_i(z_i) \geq u_i(z'_i) \).

Statements (ii) and (iii) in Theorem 1 follow from Lemmas 1-4. Statement (i) follows from (iii) and the fact that for each \( u \in \mathcal{Q}, W_c(u) \) is a compact set.\(^{11}\)

Let \( i \in N \) and \( f \) be an scf. We say that \( f \) is an \( i \)-optimal eic-scf if \( f \) selects for each \( u \in \mathcal{Q} \) an allocation that maximizes \( u_i \) in \( W_c(u) \). For each \( u \in \mathcal{Q} \) and each \( i \)-optimal scf, \( f \), agent \( i \)'s true utility \( u_i \) is a dominant strategy in \((N, \mathcal{Q}, f, u)\) [18, 14]. However, generically agent \( i \) may benefit by lying and enforcing non-cooperative outcomes that cannot be sustained by her true report in this game. Indeed, this is true for all eic-scfs.

**Proposition 1.** Let \( i \in N \) and \( f \) be an eic-scf. The set of profiles \( u \in \mathcal{Q} \) for which \( \max\{u_i(z_i) : z \in \mathcal{O}(N, \mathcal{Q}, f, u)\} \) is greater than max\{\( u_i(z_i) : z \in \mathcal{O}(N\setminus\{i\}, \mathcal{Q}, f, u)\)\} contains an open dense set of \( \mathcal{Q} \).\(^{12}\)

**Proof.** Let \( i \in N \) and \( f \) be as in the statement of the proposition. By Theorem 1, for each \( u \in \mathcal{Q} \), \( \max\{u_i(z_i) : z \in \mathcal{O}(N, \mathcal{Q}, f, u)\} \) is greater than or equal to \( \max\{u_i(z_i) : z \in \mathcal{O}(N \setminus \{i\}, \mathcal{Q}, f, u)\} \). Suppose that these two expressions are equal for \( u \). Let \( z \in \arg\max\{u_i(z_i) : z \in \mathcal{O}(N \setminus \{i\}, \mathcal{Q}, f, u)\} \). By Theorem 1, \( z \in W_c(u) \). Thus, \( u_i(z_i) = \max\{u_i(z_i) : z \in \mathcal{O}(N, \mathcal{Q}, f, u)\} \). Thus, \( z \) is the best allocation in \( W_c(u) \) for agent \( i \). Thus, for each \( j \in N \setminus \{i\}, j \geq (u, z) i \) [17, 18].

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\(^{11}\)One can also construct such an equilibrium as follows. Let \( i \in N \setminus T \). If \( z \in W_c(u) \) is the best allocation in \( W_c(u) \) for agent \( i \), then for each \( j \in N \setminus \{i\}, j \geq (u, z) i \) [17, 18]. Lemma 3 completes the argument.

\(^{12}\)Here we are identifying \( \mathcal{Q} \) with \( \mathbb{R}^{n-1} \).
Lemma 2, there is some \( j \in N \setminus \{i\} \), such that \( i \succeq (u, z) j \). Thus, there is a list of different agents, \( i_1, i_2, \ldots, i_k \) such that \( k > 1 \), \( u_i(z_{i_1}) = u_i(z_{i_2}), \ldots, u_i(z_{i_k}) = u_i(z_{i_k}) \), and \( u_i(z) = u_i(z_{i_k}) \). For simplicity, denote \( i_1 \equiv 1, i_2 \equiv 2 \) and so on. Then, \( x_1 + v_{\mu_1}^1 = v_{\mu_1}^2 + \ldots + v_{\mu_k}^k = x_k + v_{\mu_k}^k \) and \( x_k + v_{\mu_k}^k = x_1 + v_{\mu_1}^1 \). Adding up these \( k \) equations we obtain that \( u_1^1 + \ldots + u_k^1 = u_1^2 + \ldots + u_k^2 \). Then, \( u \) belongs to the union of a finite family of hyperplanes in \( \mathbb{R}^{N-1} \times \alpha \). The set of profiles \( u \in \mathcal{O}^N \) for which \( \max \{ u_i(z_i) : z \in \mathcal{O}(N, \mathcal{O}^N, f, u) \} \) is greater than \( \max \{ u_i(z_i) : z \in \mathcal{O}(N \setminus \{i\}, \mathcal{O}^N, f(u_{i', i'}), u_{N \setminus \{i\}}) \} \) contains the complement of this finite family of hyperplanes, which is an open dense set of \( \mathcal{O}^N \).

5 Discussion and concluding remarks

If the domain of admissible preferences is not restricted to be quasi-linear, there might be limit equilibrium outcomes from the manipulation of an eic-scf that are not eic allocations for the true preferences. Indeed, there might be inefficient limit equilibrium outcomes.

Consider the domain of preferences represented by utility functions \( u \equiv (u_i)_{i \in N} \) satisfying two properties: (i) money-monotonicity, i.e., for each \( \alpha \in A \) and each \( \{x_\alpha, x'_\alpha \} \subseteq \mathbb{R} \) such that \( x'_\alpha > x_\alpha \), \( u_i(x'_\alpha, \alpha) > u_i(x_\alpha, \alpha) \), and (ii) no object is infinitely better than any other, i.e., for each \( \{\alpha, \beta\} \subseteq A \) and each \( x_\beta \in \mathbb{R} \), there is \( x_\alpha \in \mathbb{R} \) such that \( u_i(x_\beta, \beta) = u_i(x_\alpha, \alpha) \). These two properties jointly imply continuity, i.e., weak upper and weak lower contour sets are closed in the product topology on \( \mathbb{R} \times A \). Thus, our assumption of existence of utility representation is without loss of generality. Moreover, we can assume that utility functions are continuous. Let \( \mathcal{U} \) be the domain of these admissible utility functions. All our definitions in Sec. 3 generalize when one considers the admissible domain of preferences to be \( \mathcal{U} \). Most importantly, limit equilibria of \( (S, \mathcal{U}, f(u_{i', i'}), u_N) \) defined with respect to a continuous utility representation of preferences coincides with utility-free limit equilibria as defined in Remark 1.13

Example 1. Let \( N \equiv \{1, 2, 3\} \), \( A \equiv \{\alpha, \beta, \gamma\} \), and \( M \equiv 0 \). Let \( g \) be an eic-scf that assigns to each \( u \in \mathcal{U}^N \) one of the best allocations for agent 2 that assigns object \( \alpha \) to agent 1 whenever possible. Fix \( \delta > 0 \). Let \( u \in \mathcal{U}^N \) be a utility profile with indifference sets as in Fig. 2 (a). Let \( z \in Z \) be the allocation at which \( z_1 \equiv (0, \alpha), z_2 \equiv (0, \beta), \) and \( z_3 \equiv (0, \gamma) \). Since \( u_1(z_2) > u_1(z_1) \), we have that \( z \not\in \mathcal{W}_e(u) \). Moreover, since \( u_2(\frac{\delta}{2}, \alpha) > u_2(z_2), z \) is not Pareto efficient for \( u \).

We claim that for each \( \varepsilon > 0 \), \( z \) is the outcome of a utility-free \( \varepsilon \)-Nash equilibrium of \( (N, \mathcal{U}, g, u) \). Let \( u' \) be a utility profile with indifference sets shown in Fig. 2 (b); \( u'_e \) is quasi-linear. Notice that both agents 1 and 3, under preferences \( u' \), are indifferent among all three bundles \( z_1, z_2 \) and \( z_3 \). Agent 2 strictly prefers \( z_2 \) to both \( z_1 \) and \( z_3 \) under preferences \( u'_e \). Thus, \( z \in \mathcal{W}_e(u'), 1 \succeq (u', z)2, \) and \( 3 \succeq (u', z)2 \). By [18, Proposition 1], \( z \in \arg\max\{u'_e : \mathcal{W}_e(u')\} \). Thus, \( g(u') = z \).

\(^{13}\)an earlier version of this paper, whose abstract was published in EC’13 proceedings, wrongly stated that Theorem 1 holds when the admissible domain of preferences is \( \mathcal{U} \).
Now, since agent 2 also strictly prefers $z_2$ to both $z_1$ and $z_3$ under the true preferences $u_3$, then $z \in W_e(u^*_2, u_2)$, $1 \succeq (u^*_2, u_2, z)2$, and $3 \succeq (u^*_2, u_2, z)2$. Thus, $z \in \text{argmax} \{u_2 : W_e(u^*_2, u_2)\}$. By Theorem 2, $u^*_2$ is a best response for agent 2 to $u^*_2$. Let $z'$ be the allocation such that $z'_1 \equiv (\frac{2}{3}, \alpha)$, $z'_2 \equiv (-\frac{2}{3} - 2\delta, \beta)$, and $z'_3 \equiv (2\delta, \gamma)$. Observe that $u_1(z'_1) > \max\{u_1(z'_3), u_1(z'_2)\}$; $u_2(z'_2) = u_2^*(z'_1) = u_2(z'_3)$; and $u_3(z'_3) = u_3^*(z'_1) > u_3^*(z'_2)$. Thus, $z' \in W_e(u^*_2, u_2)$, $2 \succeq (u^*_2, u_1, z')1$, and $3 \succeq (u^*_2, u_1, z')1$. Thus, $\{z'\} = \text{argmax} \{u_1 : W_e(u^*_2, u_1)\}$. By Theorem 2, for each $u'_1 \in \mathcal{U}$, $u_3(z'_1) \geq u_1(g(u^*_2, u'_1))$. Thus, by changing her report at $u'^*$, agent 1 cannot achieve an allocation that is preferred to $(\frac{2}{3}, \alpha)$, i.e., the bundle obtained by adding $\frac{2}{3}$ of money to her consumption at $z$. Finally, let $z$ be the allocation such that $z_1 \equiv (\epsilon, \alpha)$, $z_2 \equiv (-2\epsilon, \beta)$, and $z_3 \equiv (\epsilon, \gamma)$. Observe that $u_1^*(z_1) = u_1(z_3) > u_2^*(z_2) = u_2(z_1) > u_2(z_3)$; and $u_3(z_3) > \max\{u_3(z_1), u_3(z_2)\}$. Thus, $z \in W_e(u^*_3, u_3)$ and $2 \succeq (u^*_3, u_3, z)1 \succeq (u^*_3, u_3, z)3$. Thus, $\{z\} = \text{argmax} \{u_3 : W_e(u^*_3, u_3)\}$. By Theorem 2, for each $u'_2 \in \mathcal{U}$, $u_3(z'_3) \geq u_3(g(u^*_3, u'_2))$. Thus, by changing her report at $u'^*$, agent 3 cannot achieve an allocation that is preferred to $(\epsilon, \gamma)$, i.e., the bundle obtained by adding $\epsilon$ of money to her consumption at $z$. Thus, $u'^*$ is a utility-free $\epsilon$-Nash equilibrium of $(N, \mathcal{U}^N, g, u)$. Moreover, $z = g(u'^*)$. Thus, $z \in \partial(N, \mathcal{U}^N, g, u)$.

![Figure 2](image_url)

**Figure 2:** Each point $y$ on the axis corresponding to an object, say $a$, represents the corresponding bundle $(y, a)$. Segments connect bundles that are indifferent for $u_i$ ("indifference curve"); (a) illustrates indifference sets for utility profile $u$; (b) illustrates indifference sets for utility profile $u'$, which is a utility-free $\epsilon$-equilibrium of $(N, \mathcal{U}^N, g, u)$.

Finally, we discuss the extension of our analysis to other resource allocation environments. Let $f$ be a eic-scf. We showed that when preferences are quasi-linear, the outcomes from the manipulation of $f$ when a set of agents, say $T \subseteq N$, are sincere is a subset of its manipulation outcomes when all agents are strategic. In general, each non-cooperative outcome that can be sustained by truthful reports for agents in $T$ when all agents are strategic is a non-cooperative outcome when agents in $T$ are sincere. The remarkable feature of our environment is that strategic agents non-cooperatively coordinate only on outcomes that can be sustained as equilibrium outcomes when all agents are strategic and agents in $T$ report their true preferences. This phenomenon is not exclusive of our environment. In the exchange of goods among two agents with classical preferences, it is well known that the non-cooperative outcomes from the manipulation of a Walrasian scf is the lens shaped area in which each agent receives a convex combination of her initial endowment and a point in her offer curve [34, 35]. Suppose that there are two agents
N \equiv \{1, 2\}, agent 1 is sincere, and agent 2 is strategic. If a Walrasian scf is operated here, agent 2’s attainable allocations are those in agent 1’s offer curve. Thus, the outcome will be agent 2’s preferred allocation in agent 1’s offer curve. Moreover, this outcome can be sustained as a non-cooperative outcome when both agents are strategic and agent 1 reports her true preferences. It is an open question to determine whether this is a general result for more than two agents when an arbitrary set of agents are sincere. This two-agent example also illustrates that the presence of sincere agents may not correct the inefficiency induced by agents’ manipulation for Pareto efficient scfs whose manipulation leads to inefficient equilibria: agent 2’s preferred allocation in agent 1’s offer curve may be inefficient. It is not common that, as in our environment, the manipulation of scfs produces only normatively compelling allocations.\footnote{In classical exchange economies with transferable utility, the manipulation of scfs that are Pareto efficient, individually rational, and monotonic in the sense of \cite{36} is essentially the set of manipulation outcomes of the Walrasian scfs. Moreover, the manipulation of a general family of scfs defined by means of cooperative bargaining solutions containing the Nash solution and the Kalai-Smorodinsky solution may produce inefficient allocations as well \cite{37}.}

There are some other exceptions, however. In a classical exchange economy, an allocation is “constrained” Walrasian whenever it can be sustained as a competitive equilibrium in which agents’ budget sets are constrained by feasibility.\footnote{Constrained Walrasian allocations are weakly Pareto efficient. They are Pareto efficient when preferences are strictly increasing.} The set of outcomes from the manipulation of the Shapley value scf in smooth exchange economies with transferable utility is the set of constrained Walrasian allocations for true preferences \cite{30}. The set of outcomes from the manipulation of the “equal sacrifice” scf of \cite{31} in the fair allocation of a bundle of normal goods is the set of constrained Walrasian allocations operated from equal division. Different from our environment, these scfs force strategic agents to lie. Thus, one can easily see that there is no inclusion relation between the manipulation outcomes of these scfs when all agents are strategic and when some agents are sincere (c.f., example illustrated in \cite[Fig. 1]{30}). It is an open question to determine whether the manipulation of these scfs, when some agents are sincere, produces any efficiency loss and the extent to which strategic agents may take advantage of sincere agents.

Appendix

The following results play an important role in our proofs. We state them for completeness. See the respective papers for the proof.

**Lemma 5** (Decomposition Lemma; \cite{25}). Let \( u \in \mathcal{U}^N \), \( z \equiv (x, \mu) \in W_c(u) \), and \( \bar{z} \equiv (\bar{x}, \bar{\mu}) \in W_c(u) \). Then, both \( \mu \) and \( \bar{\mu} \) are bijections between:

(i) \( \{ i \in N : u_i(z_i) > u_i(\bar{z}_i) \} \) and \( \{ a \in A : x_a > \bar{x}_a \} \).

(ii) \( \{ i \in N : u_i(z_i) = u_i(\bar{z}_i) \} \) and \( \{ a \in A : x_a = \bar{x}_a \} \).

(iii) \( \{ i \in N : u_i(\bar{z}_i) > u_i(z_i) \} \) and \( \{ a \in A : \bar{x}_a > x_a \} \).

**Theorem 2** (Theorem 1 and Remark 3 in \cite{18}, \cite{14}, \cite{33}). Let \( \mathcal{D} \subseteq \mathcal{U} \) be such that \( \mathcal{D} \) is an eic-scf, \( u \in \mathcal{D}^N \), \( i \in N \), and \( z \equiv (x, \mu) \) such that \( z \in \arg\max \{ u_i(z_i) : z \in \mathcal{D} \} \).
\(W_i(u).\) Then, for each \(u'_i \in \mathcal{U}, u_i(f_i(u_{-i}, u'_i)) \leq u_i(z_i).\) Moreover, for each \(\epsilon > 0\) there is \(u'_i \in \mathcal{U}\) such that \(u_i(f_i(u_{-i}, u'_i)) \geq u_i(x_{\mu_i} - \epsilon, \mu_i).\)

References


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URL http://dx.doi.org/10.1016/0047-2727(90)90003-Z

URL http://dx.doi.org/10.1007/s00182-009-0200-7

URL http://econtheory.org/ojs/index.php/te/article/view/20140753/0

URL http://dx.doi.org/10.1006/game.1995.1015

URL http://dx.doi.org/10.1016/j.geb.2007.01.011

URL http://dx.doi.org/10.1016/j.jet.2010.10.005


URL http://dx.doi.org/10.1016/j.jet.2007.06.006

URL http://dx.doi.org/10.1016/j.geb.2011.07.006

URL http://scholarworks.umass.edu/econ_workingpaper/175

URL http://dx.doi.org/10.1016/j.jet.2006.04.003

URL http://www.jstor.org/stable/40247645

URL http://www.jstor.org/stable/30035023
URL http://www.jstor.org/stable/2938172


URL http://www.jstor.org/stable/2297433

URL http://dx.doi.org/10.1007/BF01254542

URL http://dx.doi.org/10.1016/j.geb.2012.01.001

URL http://dx.doi.org/10.1007/BF01213627

URL http://ideas.repec.org/p/mtl/montec/04-2012.html


URL http://dx.doi.org/10.1016/0022-0531(82)90014-X

URL http://www.jstor.org/stable/2566947

URL http://www.jstor.org/stable/1911514